The Bohm Criterion
In Plasmas With One or Two Ion Species

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Introduction

Part I: One ion species

• Review the conventional kinetic (generalized) Bohm criterion (KBC)
  – Show that it is not so general as one may think

• A KBC can be formulated from positive-exponent velocity moments
  – Avoids problems that arise in the conventional KBC where a $v_z^{-1}$ moment is used

• Consider the role of ion-acoustic instabilities in the presheath
  – Show that ion-acoustic instabilities enhance collisions and can cause Maxwellization of ions and electrons near the sheath edge (if $T_e \gg T_i$)
  – Compare to LIF measurements from Claire et al POP (2006)

Part II: Two ion species

• What determines the Bohm criterion in two ion species plasmas?
  – Bohm criterion is a single constraint in two unknowns ($V_1$ and $V_2$)
  – What is the second constraint needed to uniquely determine $V_1$ and $V_2$?

• Review the discrepancy between experimental and theoretical literature

• Show that ion-ion two-stream instabilities can be important
  – Instability-enhanced collisions give a strong collisional friction
  – Constitute a stiff system that relates $V_1$ and $V_2$
2-scale theory: defining the sheath edge

- There is at least 3 length scales: sheath ($\lambda_D$), presheath ($\lambda_{i-n}$), transition region ($\lambda_D^{4/5} \lambda_{i-n}^{1/5}$)
- In the limit $\lambda_D \ll \lambda_{i-n}$, the transition region is very narrow (2 scales)

The sheath criterion (Riemann IEEE 1995, & others)

- Define sheath edge as location where quasineutrality breaks down

$$\frac{d^2 \phi}{dz^2} = -4\pi \left[ \rho(\phi = 0) + \frac{d\rho}{d\phi} \bigg|_{\phi=0} \phi + \ldots \right] \approx -4\pi \frac{d\rho}{d\phi} \bigg|_{\phi=0} \phi$$

where $\rho \equiv \sum_s q_s n_s$ is the charge density

- Multiplying by $d\phi/dz$ and integrating wrt $z$ yields

$$\frac{E^2}{4\pi} + \frac{d\rho}{d\phi} \bigg|_{\phi=0} \phi^2 = C$$

- Since $\phi \to 0$ as $z/\lambda_D \to \infty$ (on sheath length scale), $C = 0$

- The sheath criterion is then $d\rho/d\phi|_{\phi=0} \leq 0$, or

$$\sum_s q_s \frac{dn_s}{dz} \bigg|_{z=0} \geq 0$$
The original Bohm criterion

- For single species of singly-charged ions and electrons, the SC is

\[
\frac{dn_i}{dz} - \frac{dn_e}{dz} \geq 0
\]

- Bohm assumed Boltzmann distributed electrons

\[
n_e = n_o \exp\left(-\frac{e\phi}{T_e}\right)
\]

and cold ions accelerated by the presheath

\[
\frac{1}{2} M_i V_i^2 = e\phi \quad \Rightarrow \quad V_i = \sqrt{2e\phi/M_i}
\]

- Also using the continuity equation \(d(n_i V_i)/dz = 0\) gives

\[
\frac{dn_e}{dz} = -n_e \frac{e}{T_e} \frac{d\phi}{dz} \quad \text{and} \quad \frac{dn_i}{dz} = -n_i \frac{e}{M_i V_i^2} \frac{d\phi}{dz}
\]

- Putting these into the sheath criterion gives

\[
V_i \geq \sqrt{\frac{T_e}{M_i}} \equiv c_s
\]

What happens for more general ion and electron distributions?
KBC: seek generalization for arbitrary distributions

- The conventional KBC is typically cited as

\[ \frac{1}{M_i} \int_{-\infty}^{\infty} d^3v \frac{f_i(\vec{v})}{v_z^2} \leq - \frac{1}{m_e} \int_{-\infty}^{\infty} d^3v \frac{\partial f_e(\vec{v})}{\partial v_z} \]

But there are problems with this!

- Low velocity particles dominate in this equation
  - If \( f_i(v_z = 0) \neq 0 \), the ion term diverges
  - If \( \partial f_e/\partial v_z |_{v_z=0} \neq 0 \) the electron term diverges

- Example: For flowing Maxwellian ions and stationary Maxwellian electrons, the KBC gives

\[ \infty \leq n_e/T_e \]

  - The conventional KBC apparently doesn’t work for this example
  - We will show that these distribution functions can be expected

- Apparently the generalized Bohm criterion is not as general as one might think
KBC is based on the (collisionless) Vlasov equation

- Putting \( n_s \equiv \int d^3v f_s \) into the sheath criterion gives

\[
\sum_s q_s \int_{-\infty}^{\infty} d^3v \frac{\partial f_s}{\partial z} \geq 0
\]

- Start from the collisionless Vlasov equation

\[
v_z \frac{\partial f_s}{\partial z} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v_z} = 0
\]

- Step 1: Take \( v_z^{-1} \) moment and put into the sheath condition

\[
\sum_s \frac{q_s^2}{m_s} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} \frac{1}{v_z} \frac{\partial f_s}{\partial v_z} \leq 0
\]

- Step 2: Integrate the ion term by parts (single ion species)

\[
\frac{1}{M_i} \int_{-\infty}^{\infty} d^3v \frac{f_i(\vec{v})}{v_z^2} \leq -\frac{1}{m_e} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} \frac{\partial f_e(\vec{v})}{\partial v_z}
\]
Derivations of KBC make tacit assumptions on $f$

- **Issue 1**: The integration-by-parts step
- But, integration-by-parts is only valid if $\partial f_i/\partial v_z|_{v_z=0} = 0$
- The contentious step is of the form

$$
\int_{-\infty}^{\infty} dx \frac{1}{x} \frac{df}{dx} = \int_{-\infty}^{\infty} dx \frac{d}{dx} \left( \frac{f}{x} \right) + \int_{-\infty}^{\infty} dx \frac{f}{x^2},
$$

- Consider the example: $f(x) = \exp(-x^2)$
  - The left side can be evaluated directly to give a finite answer

$$
\int_{-\infty}^{\infty} dx \frac{1}{x} \frac{df}{dx} = -2 \int_{-\infty}^{\infty} dx e^{-x^2} = -2\sqrt{\pi}.
$$

  - But the right side will diverge for this example:

$$
\int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x^2} = \lim_{\epsilon \to 0} \left( \int_{-|\epsilon|}^{-|\epsilon|} dx \frac{e^{-x^2}}{x^2} + \int_{|\epsilon|}^{\infty} dx \frac{e^{-x^2}}{x^2} \right)
$$

$$
= -2\sqrt{\pi} + \lim_{\epsilon \to 0} \left( \frac{2}{|\epsilon|} e^{-|\epsilon|^2} + 2\sqrt{\pi} \text{erf}(|\epsilon|) \right)
$$

$$
= -2\sqrt{\pi} + \lim_{\epsilon \to 0} \frac{2}{|\epsilon|} e^{-|\epsilon|^2} \to \infty
$$

Effectively assumes that: $\partial f_i/\partial v_z|_{v_z=0} = 0$
Derivations of KBC make tacit assumptions on $f$

- A second (more significant) restriction arises from taking the $v_z^{-1}$ moment of the collisionless Vlasov equation.

- Consider what happens if the kinetic equation is used

$$v_z \frac{\partial f_s}{\partial z} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v_z} = C(f_s)$$

- Taking the $v_z^{-1}$ moment of this and putting the result into the sheath condition gives

$$\sum_s q_s^2 \frac{1}{m_s} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} \frac{\partial f_s}{\partial v_z} \leq \sum_s \frac{q_s}{E} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} C(f_s) \quad (1)$$

- The collision operator has the general form

$$C(f_s, f_s') = -\frac{\partial}{\partial \vec{v}} \cdot \int d^3v' \leftrightarrow Q \cdot \left( \frac{1}{m_s'} \frac{\partial}{\partial \vec{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \vec{v}} \right) f_s(\vec{v}) f_s'(\vec{v}')$$

- The collision operator term also diverges when $\frac{\partial f}{\partial v_z}|_{v_z=0} \neq 0$, or $f_s(v_z = 0) \neq 0$

- If the distribution contains slow particles, it is inconsistent to neglect the collision operator term, but not the left-hand side of (1)

- The same problems can arise from an ionization source term $C(f_s) \rightarrow S_i$
Instead, consider positive-exponent moments

- The density moment ($\int d^3v \ldots$) of the kinetic equation yields the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{V}_s) = 0$$  \hspace{1cm} (2)

in which the density and fluid flow velocity are defined as:

$$n_s \equiv \int_{-\infty}^{\infty} d^3v \ f_s \quad \text{and} \quad \vec{V}_s \equiv \frac{1}{n_s} \int_{-\infty}^{\infty} d^3v \ \vec{v}f_s$$

- The momentum moment ($\int d^3v \ m_s \vec{v} \ldots$) yields the momentum evolution equation

$$m_s n_s \left( \frac{\partial \vec{V}_s}{\partial t} + \vec{V}_s \cdot \nabla \vec{V}_s \right) = n_s q_s \vec{E} - \frac{\partial p_s}{\partial \vec{x}} - \nabla \cdot \vec{\pi}_s + \vec{R}_s.$$  \hspace{1cm} (3)

in which the scalar pressure, stress tensor, temperature and friction force density are defined as:

$$p_s \equiv \int_{-\infty}^{\infty} d^3v \ \frac{1}{3} m_s \vec{v}_r^2 f_s = n_s T_s, \quad \vec{\pi}_s \equiv \int_{-\infty}^{\infty} d^3v \ m_s \left( \vec{v}_r \vec{v}_r - \frac{1}{3} \vec{v}_r^2 \ I \right) f_s$$

$$T_s \equiv \frac{1}{n_s} \int_{-\infty}^{\infty} d^3v \ \frac{1}{3} m_s \vec{v}_r^2 f_s = \frac{1}{2} m_s v_{Ts}^2$$ \quad and \quad $\vec{R}_s \equiv \int_{-\infty}^{\infty} d^3v \ m_s \vec{v} C(f_s)$

here $\vec{v}_r \equiv \vec{v} - \vec{V}_s$
A KBC from positive-exponent moments

- Putting the 1-D continuity equation
  \[ n_s \frac{dV_{z,s}}{dz} + V_{z,s} \frac{dn_s}{dz} = 0 \]
  into the sheath condition gives
  \[ \sum_s q_s n_s \frac{dV_{z,s}}{dz} \bigg|_{z=0} \leq 0 \]

- Solving \( dV_{z,s}/dz \) from the momentum balance
  \[ m_s n_s V_{z,s} \frac{dV_{z,s}}{dz} = n_s q_s E - \frac{dp_s}{dz} - \frac{d\Pi_{zz,s}}{dz} + R_{z,s} \]
  and putting it in above gives a Bohm criterion
  \[ \sum_s q_s \left[ q_s n_s - \left( n_s \frac{dT_s}{dz} + \frac{d\Pi_{zz,s}}{dz} - R_{z,s} \right) / E \right] \leq 0 \]

- At sheath edge: \( (...) / E \sim \lambda_{De} / \lambda_i \ll q_s n_s \), so:
  \[ \sum_i \frac{q_i^2}{e^2 n_e V_{z,i}^2} \frac{c_{s,i}^2}{V_{z,i}^2 - v_{T,i}^2 / 2} \bigg|_{z=0} \leq 1 \]
If $f_i$ has no slow particles, get same criterion

- **Example 1: consider Bohm’s approximation**
  - Monoenergetic ions:
    \[ f_i = n_i \delta(\vec{v} - \vec{V}_i) \]
  - Maxwellian electrons
    \[ f_e = \frac{n_e}{\pi^{3/2}v_{Te}^3} \exp\left(-\frac{v^2}{v_{Te}^2}\right) \]

- In this case, both theories give: \[ V_i \geq c_s = \sqrt{\frac{T_e}{M_i}} \]

- The conventional KBC works in this case because
  \[ f_i(v_z = 0) = 0 \]
  and
  \[ \left. \frac{\partial f_e}{\partial v_z} \right|_{v_z=0} = 0 \]
  at the sheath edge

- If these conditions are not met, the conventional KBC has divergent integrals
Some ionization sources are conveniently chosen

- Example 2: Tonks-Langmuir plasma: collisionless ions with ionization
- Recall that conventional KBC assumes $C(f_s)|_{v_z=0} = 0$, or $S_i|_{v_z=0} = 0$
- Emmert source function has a “flux” form:
  \[ S_i = h(z) \frac{v_z}{v_{Ti}^2} \exp\left(-\frac{v_z^2}{v_{Ti}^2}\right) \]
- Conveniently chosen so $S_i(v_z = 0) = 0$
- Then $f_i(v_z = 0) = 0$ @ sheath edge, and the conventional KBC is satisfied

A Maxwellian source function seems more physical

- It is the background neutrals that are ionized
- These are in thermal equilibrium – Maxwellian
- Source should also be Maxwellian
Conventional KBC diverges for a Maxwellian source

- Bissell and Johnson, Phys. Fluids (1987) used a Maxwellian:
  \[ S_i = \frac{h(z)}{v_T} \exp(-v_z^2/v_T^2) \]
- Thus, \( S_i/v_z |_{v_z=0} \to \infty \)
- Sheridan, Phys. Plasmas (2001) showed with simulations that \( f_i(v_z = 0) \neq 0 \) @ sheath edge
- Since \( f_i(v_z = 0) \neq 0 \), the conventional KBC diverges
- New KBC gives \( V_i \gtrsim c_s \)
- No divergences in the positive-exponent velocity moments
  Sheridan’s simulation data shows IVDF for different source temps →
LIF measurements show 3 regions in the presheath

(a) Maxwellian in the bulk plasma and entrance to presheath (collisional)

(b) A 2-step distribution in the mid presheath (collisionless)

(c) Maxwellian at the presheath exit and in sheath (collisional)

- Conventional KBC gives $\infty \leq n_e/T_e$, new KBC gives $V_i \geq c_s$

Why are ions collisional near the sheath edge?

- The ion-ion collision frequency is proportional to $1/\bar{v}^3$:
  \[
  \nu_{i-i}^{LB} = \frac{2\pi n_i q_i^4}{m_i^2 \bar{v}^3} \ln \Lambda
  \]

- Collision length $\propto \bar{v}^4$: $\lambda_{i}^{i-i} \approx \bar{v}/\nu_{i}^{i-i} \propto \bar{v}^4$

  Consider the plasma parameters of Claire et al.:
  $\text{Ar}^+, T_i = 0.027 \text{ eV}, T_e = 2.5 \text{ eV}, n_e = 5.5 \times 10^9 \text{ cm}^{-3}, p_n = 1.8 \times 10^{-4} \text{ mbar}$

- For these: $\lambda_{i}^{i-i} \approx 1.8 \times 10^{-13} \bar{v}^4 \text{ cm}$

- Bulk plasma: $\bar{v} \approx v_{Ti} \approx 3.4 \times 10^2 \text{ m/s} \Rightarrow \lambda_{i}^{i-i} \approx 2.4 \text{ mm}$

- Presheath: $\bar{v} \approx V_i$, so near sheath $\bar{v} \approx c_s = 2.4 \times 10^3 \text{ m/s} \Rightarrow \lambda_{i}^{i-i} \approx 6.0 \text{ m}$

- Presheath length $\approx \lambda_{i-n}^{i-n} \approx 23 \text{ cm}$

- Somewhere in presheath ions become collisionless: $\lambda_{i}^{i-i} \lesssim \lambda_{i-n}^{i-n}$

- Expect Maxwellian in entrance to presheath, but 2-step as the sheath is approached – like simulations, but not what is measured
Kinetic theory includes linear instabilities in $C(f)$

- The kinetic equation $\frac{df_s}{dt} = \sum_{s'} C(f_s, f_{s'})$
  has a collision operator of the Landau form:

  $$ C(f_s, f_{s'}) = -\frac{\partial}{\partial \vec{v}} \cdot \int d^3v' \leftrightarrow Q(\vec{v}, \vec{v}') \cdot \left( \frac{1}{m_s} \frac{\partial}{\partial \vec{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \vec{v}} \right) f_s(\vec{v}) f_{s'}(\vec{v}') $$

- The collisional kernel is the sum of two terms: $\leftrightarrow Q = \leftrightarrow Q_{LB} + \leftrightarrow Q_{IE}$

  Lenard-Balescu term describes stable Coulomb interactions

  $$ \leftrightarrow Q_{LB} = \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{k \cdot \vec{k}}{k^4} |\hat{\varepsilon}(\vec{k}, \vec{k} \cdot \vec{v})|^2 $$

  NEW: Instability-enhancements describe interactions with fluctuations

  $$ \leftrightarrow Q_{IE} = \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\vec{k} \cdot \vec{k}}{k^4} \sum_j \gamma_j e^{2\gamma_j t} $$

  $$ \omega_{R,j} is the real part and \gamma_j the imaginary part of the j^{th} unstable mode

  $$ \hat{\varepsilon}(\vec{k}, \omega) = 1 + \frac{4\pi}{k^2} \sum_s q_s^2 \int d^3v \frac{\vec{k} \cdot \partial f_s(\vec{v})/\partial \vec{v}}{\omega - \vec{k} \cdot \vec{v}} $$

  is the plasma dielectric function

  Baalrud, Hegna, Callen, POP (2008) and (2010)
Discrete particles are the source of fluctuations

- The theory is similar to quasilinear theory

\[
\frac{df_s}{dt} = \frac{\partial}{\partial \vec{v}} \cdot \mathcal{D}_v \cdot \frac{\partial f_s}{\partial \vec{v}}
\]

where the velocity-space diffusion coefficient is

\[
\mathcal{D}_v = \frac{q_s^2}{m_s^2} 8\pi \sum_j \int d^3k \frac{\vec{k} \cdot \vec{v}}{k^4} \frac{\gamma_j \mathcal{E}_j(\vec{k})}{(\omega_{R,j}^2 - \vec{k} \cdot \vec{v})^2 + \gamma_j^2}
\]

and the spectral energy density is

\[
\mathcal{E}_j^{ql}(\vec{k}) = \frac{|\delta \hat{E}(\vec{k}, t = 0)|^2 e^{2\gamma_j t}}{(2\pi)^3 V} \frac{\gamma_j}{8\pi}
\]

- BUT...the discrete particle source of fluctuations self consistently determines \( \mathcal{E} \) in the new kinetic theory

\[
\mathcal{E}_j^{\text{kin}}(\vec{k}) = \sum_{s'} \frac{q_{s'}^2}{4\pi^2 |\partial \hat{\varepsilon} / \partial \omega|^2_{\omega_j}} \int d^3v' \frac{f_{s'}(v') e^{2\gamma_j t}}{(\omega_{R,j} - \vec{k} \cdot \vec{v}')^2 + \gamma_j^2}
\]

- In conventional quasilinear theory the spectral energy density \( \mathcal{E}(\vec{k}) \) is an input – it must be determined external to the theory
- Kinetic theory can also describe component collisions \( C(f_s, f_{s'}) \)
- As long as \( \gamma / \omega_R \ll 1 \), the unique equilibrium is Maxwellian
Ion-acoustic instabilities can enhance collisions

- Microinstabilities lead to enhanced collisions [Baalrud, Hegna, Callen POP (2008), (2010)]

\[
\nu_{IE}^{s-s'} \approx \frac{2 n_s q_s^2 q_s^2}{m_s^2 \bar{v}^2} \int d^3k \frac{\bar{k} k}{k^4} \gamma \exp(2 \gamma t) \frac{[\omega_R - \bar{k} \cdot \bar{v} + \gamma^2][\omega_R - \bar{k} \cdot \bar{v}'] + \gamma^2]}{[\omega_R - \bar{k} \cdot \bar{v}] + \gamma^2}] |\partial \hat{\epsilon}(\bar{k}, \omega)/\partial \omega|_{\omega_R}^2
\]

- As long as \( \gamma/\omega_R \ll 1 \Rightarrow \) enhanced collisions lead to unique Maxwellian distributions

- The ion-acoustic dispersion relation is

\[
\omega_{\pm} = \left( \bar{k} \cdot \bar{V}_i \pm \sqrt{\frac{n_i}{n_e}} \frac{kc_s}{\sqrt{1 + k^2 \lambda_{De}^2}} \right) \left( 1 \pm i \sqrt{\frac{n_i}{n_e}} \frac{\sqrt{\pi m_e/8M_i}}{(1 + k^2 \lambda_{De}^2)^{3/2}} \right)
\]

- For the ion-acoustic instability, the enhanced scattering frequency is

\[
\nu_{IE}^{s-s} \approx \frac{\nu_{LB}^{s-s} 1 + \kappa_c^2}{8 \ln \Lambda (1 + \kappa_c^2)^2} \exp \left( \sqrt{\frac{\pi m_e n_i}{16M_i n_e \lambda_{De}}} Z \right)
\]

Here \( Z = 0 \) is at the presheath-bulk boundary and

\[
\kappa_c \equiv \begin{cases} 
\sqrt{c_s^2/V_i^2 - 1}, & \text{for } V_i \leq c_s \\
0, & \text{for } V_i \geq c_s
\end{cases}
\]
Consideration of IE collisions predicts the 3 regions
Part II: Two ion species

- For 2 species, the Bohm criterion admits an infinite number of solutions

\[
\frac{n_1 c_{s,1}^2}{n_e V_1^2} + \frac{n_2 c_{s,2}^2}{n_e V_2^2} = 1 \quad (\text{for } T_i/T_e \ll 1)
\]

- What is the second condition needed to uniquely determine \(V_1\) and \(V_2\)?

- If ions are collisionless \(V_i = \sqrt{2e|\phi_{ps}|/M_i}\), thus \(V_1/V_2 = \sqrt{M_2/M_1}\)

- Putting this into the Bohm criterion gives:

  “Individual” sound speed: \(V_i = c_{s,i} = \sqrt{\frac{T_e}{M_i}}\)

- Franklin (J. Phys. D: Appl. Phys. 33, 3186 (2000)) has also accounted for ion-neutral collisions
  - Typically don’t significantly modify speeds at the sheath edge (more drag ⇒ more \(E\))
  - Only if the ion-neutral collision frequencies are very different for each species do significant deviations from the individual sound speed occur
  - This is rare for the noble gases typical of LTP experiments
  - Ion-ion collisions are assumed weak (and are if the plasma is stable)
Experiments do not agree with previous theory

- Experiments measure a near common speed

  “System” sound speed: \( V_i = c_s = \sqrt{\sum_i \frac{n_i}{n_e} c_{s,i}^2} \)

- Cold ion temperature regime: \( T_e/T_i \sim 50 \)
- Speeds may differ from common by \( O(v_{Ti}) \)

Experiments hint at importance of ion-ion friction

- However, ion-ion friction is weak if the plasma is stable
- Recall, the collisional friction is
  \[
  \vec{R}^{s/s'} = \int d^3 \vec{v} \; m_s \vec{v} C(f_s, f_s') = \vec{R}_{LB}^{s/s'} + \vec{R}_{IE}^{s/s'}
  \]
- For Maxwellian ions with \( T_s = T_s' \) stable plasma contribution is
  \[
  \vec{R}_{LB}^{s/s'} = -\frac{\sqrt{\pi}}{2} n_s m_s \nu_s \frac{\bar{v}_T^3 \Delta \vec{V}}{\Delta V^4} \psi \left( \frac{\Delta V^2}{\bar{v}_T^2} \right)
  \]
  in which \( \Delta \vec{V} \equiv \vec{V}_s - \vec{V}_s' \) is the relative flow speed, \( \bar{v}_T^2 \equiv v_{Ts}^2 + v_{Ts'}^2 \),
  \[
  \psi(x) = 2/\sqrt{\pi} \int_0^x dt \; \sqrt{t} e^{-t}
  \]
is the Maxwell integral, and a reference collision frequency is
  \[
  \nu_s \equiv \frac{8\sqrt{\pi} q_s^2 q_s'^2 n_s' \ln \Lambda}{m_s^2 v_{Ts}^2 \bar{v}_T}
  \]
- For typical low-temperature plasmas (this one in particular):
  \( R_{LB}^{1-2} \approx 10 \times \) smaller than other term in momentum balance
- Need \( R_{IE}^{1-2} / R_{LB}^{1-2} \gtrsim 10 \) for instability-enhanced friction to matter
Two-stream instabilities in the cold ion limit

- Approximate dispersion relation for ion-ion two-stream instabilities:

\[ \omega = \vec{k} \cdot \left( \frac{n_2 c_s^2}{n_e c_s^2} \vec{V}_1 + \frac{n_1 c_s^2}{n_e c_s^2} \vec{V}_2 \right) + i \frac{\vec{k} \cdot \Delta \vec{V}}{1 + \alpha} \sqrt{1 - \frac{(\vec{k} \cdot \Delta \vec{V})^2}{k^2 \Delta V_{up}^2}} \left(1 + k^2 \lambda_{De}^2 \right) \]

where \( \alpha = \sqrt{n_1 M_2 / (n_2 M_1)} \) and \( \Delta V_{up}^2 = c_s^2 \left[1 + \sqrt{1 + 32 \alpha / (1 + \alpha)^2}\right] \) (decoupling of ion beams)

- Plot shows numerical (solid), quadratic (dashed), approximate (dotted)
Two-stream instabilities enhance ion-ion friction

- Calculate the instability-enhanced collisional friction: \( \vec{R}_{1E}^{s-s'} \)
- Assume \( v_T \ll c_s \sim V_s \), and flowing Maxwellian ion distributions
- Friction due to instability-enhanced interactions:

\[
\vec{R}_{1E}^{1-2} \simeq n_1 m_1 \nu_{12} \Delta \vec{V} \exp \left( \frac{\sqrt{\alpha} A}{(1 + \alpha)} \frac{\Delta V^2}{v_g \Delta V_{up} \lambda_{De}} \frac{z}{\nu_{12}} \right)
\]

in which \( A = \Delta V_{up}^2 / \Delta V^2 - 1 \),

\[
\nu_{12} = \nu_s \frac{3}{160 \sqrt{\pi}} \frac{\bar{v}_T \Delta V^4}{\Delta V_{up} c_s^4} \frac{A^{7/2}}{4 + A^{3/2}} \frac{\alpha^{5/2}(1 + \alpha^{1/3})^2}{\alpha^2 - 1}
\]

is a characteristic frequency, and

\[
v_g = \frac{n_2 c_{s2}^2 V_1 + n_1 c_{s1}^2 V_2}{n_e c_s^2}
\]

is the group speed of the waves
Two-stream instabilities ⇒ rapid/strong friction

- Get $10 \times$ enhancement within approximately $z/\lambda_{De} = 5$
- Presheath length: $l \sim 10^3 \lambda_{De} \gg$ growth length for $\vec{R}_{IE}^{1-2}$ to dominate
- Very stiff system! Friction is huge if unstable
Need to account for finite $T_i$

- So far, assumed $T_i = 0$ in which case instability for $V_1 - V_2 = \Delta V \geq 0$
- Using $V_1 - V_2 = 0$ in the Bohm criterion gives system sound speed
  
  \[ V_1 = V_2 = \sqrt{\frac{n_1}{n_e} c^2_{s,1} + \frac{n_2}{n_e} c^2_{s,2}} = c_s \]

- This agrees with previous measurements [within $O(v_{Ti})$], but ions were room temperature (cold)
- Finite $T_i$ gives stabilization for $\Delta V \leq \Delta V_c = O(v_{Ti})$ – important!
- $\Delta V_c$ is determined from the dielectric function (assume Maxwellians)

\[
1 + k^2 \chi^2_{De} = \frac{1}{2} \frac{n_1 T_e}{n_e T_1} Z' \left( \frac{k \cdot \Delta \vec{V} (\Omega - 1/2)}{k v_{T_1}} \right) + \frac{1}{2} \frac{n_2 T_e}{n_e T_2} Z' \left( \frac{k \cdot \Delta \vec{V} (\Omega + 1/2)}{k v_{T_2}} \right)
\]

where $Z$ is the plasma dispersion function

[here $\Omega$ is defined by $\omega = \frac{1}{2} \vec{k} \cdot (\vec{V}_1 + \vec{V}_2) + \vec{k} \cdot \Delta \vec{V} \Omega$]

- We will find that $\Delta V_c$ depends on relative concentrations ($n_1/n_e$) too
- This will provide a convenient way to test the theory experimentally
A fluid approximation for $\Delta V_c$

- If $v_{T1}/v_{T2} \gg 1$ or $v_{T1}/v_{T2} \ll 1$ the PDFs are separated ($\Omega_o = \Omega \pm 1/2$)

In this case, the fluid limit of $Z'$ is a good approximation

$$\hat{\epsilon} = 1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{\omega_{p1}^2}{(\omega - \vec{k} \cdot \vec{V}_1)^2 - k^2 v_{T1}^2 / 2} - \frac{\omega_{p2}^2}{(\omega - \vec{k} \cdot \vec{V}_2)^2 - k^2 v_{T2}^2 / 2}$$

which gives the instability criterion $\Delta V > (k/k_{||})\Delta V_c^{fl}$ where

$$\Delta V_c^{fl} = \sqrt{\frac{1 + \alpha}{2\alpha}} \sqrt{v_{T1}^2 + \alpha v_{T2}^2} \quad \text{in which} \quad \alpha \equiv \frac{n_1 M_2}{n_2 M_1}$$
A kinetic approximation for $\Delta V_c$

- For $v_{T1}/v_{T2} \sim 1$, the PDFs overlap a lot ($\Omega_o = \Omega \pm 1/2$)

- In this case, expanding $Z'(w)$ about $w = \pm 3/2$ is more reasonable
- Doing so leads to the instability criterion $\Delta V > (k/k_{\parallel})\Delta V_{c}^{\text{kin}}$ where

$$\Delta V_{c}^{\text{kin}} = -\frac{3}{2}|v_{T2} - v_{T1}| + \sqrt{\frac{1}{2} \left( 1 + \frac{n_2 T_1}{n_1 T_2} \right)} \left( v_{T1}^2 + \frac{n_1 T_2}{n_2 T_1} v_{T2}^2 \right)$$
Ion-ion friction can determine the Bohm criterion

- If $\Delta V_c > |c_{s1} - c_{s2}|$, no instabilities are expected and Franklin’s solution of individual sound speeds should hold.

- A condition relating the flow speed of each species is then

$$V_1 - V_2 = \Delta V_c \equiv \begin{cases} \Delta V^\text{fl}_c, & \text{or } \Delta V^\text{kin}_c \text{ if } \leq |c_{s1} - c_{s2}| \\ c_{s1} - c_{s2} & \text{if } > |c_{s1} - c_{s2}| \end{cases}$$

- Use $\Delta V^\text{fl}_c$ for $v_{T1}/v_{T2} \gtrsim 4$ or $\lesssim 1/4$ and $\Delta V^\text{kin}_c$ for $1/4 \gtrsim v_{T1}/v_{T2} \gtrsim 4$

- Putting $\Delta V = \Delta V_c$ into the Bohm criterion gives

$$\frac{n_1 c^2_{s1}}{n_e V_1^2} + \frac{n_2 c^2_{s2}}{n_e (V_1 - \Delta V_c)^2} = 1,$$

which is a quartic equation to solve for $V_1$ [only one positive real solution]

- If $\Delta V_c \ll |c_{s1} - c_{s2}|$ a handy approximate expression is

$$V_1 \simeq c_s + \frac{n_2 c^2_{s2}}{n_e c^2_s} \Delta V_c \quad \text{and} \quad V_2 \simeq c_s - \frac{n_1 c^2_{s1}}{n_e c^2_s} \Delta V_c$$

- Speeds differ from $c_s$ by an amount $\mathcal{O}(v_{Ti})$
Preasheath can be collisional near the sheath

- If instability-enhanced friction is present, the presheath has two regions
- $\Delta V$ is “locked” in the unstable region – seen in measurements

\[ V_1 = c_s + \frac{n_2}{n_e} c_s^2 \Delta V_c \]
\[ V_2 = c_s - \frac{n_1}{n_e} c_s^2 \Delta V_c \]
Ar\(^{+}\)-Xe\(^{+}\) data agrees with $\Delta V_c^{\text{kin}}$ predictions

- Data from Yip, Hershkowitz, Severn, PRL 104, 225003 (2010)
- $T_e = 0.7$ eV, $T_1 \approx T_2 = 0.04$ eV, 1 labels Ar\(^{+}\), 2 labels Xe\(^{+}\)
- Since $v_{T_1}/v_{T_2} = 1.8 \sim 1$, use $\Delta V_c = \Delta V_c^{\text{kin}}$
- Solid lines show full solution of quartic, dashed the handy formula
**He\(^+\)-Xe\(^+\) data agrees with \(\Delta V^\text{fl}_c\) predictions**

- Data from Hershkowitz, Yip, Severn, POP (in press).
- \(T_e = 1 \text{ eV}, T_1 = 0.07, T_2 = 0.04 \text{ eV}, 1\) labels He\(^+\), 2 labels Xe\(^+\)
- Since \(v_{T2}/v_{T1} \approx 6\), use \(\Delta V_c = \Delta V^\text{fl}_c\)
- Solid lines show full solution of quartic, dashed the handy formula
Why don’t I see IE collisions in my PIC simulations?

• **Thing 1:** Need to have a Coulomb collision routine in the code
  - Many LTP PIC simulations don’t include a Coulomb collision routine
  - This can be done using Monte Carlo techniques (typically used for ion-neutral)

• To get Coulomb collisions “self consistently” would need
  1. Grid resolution of interparticle distance (so macro-fields on the grid resolve single particle fields)
  2. A macroparticle density that satisfies \( n\lambda_D^3 \) = experiment value (so the collective effects of screening are accounted for)

• Of course, these cannot be satisfied (so PIC uses MCC routines)

• **Thing 2:** Probably need a modified cross section for IE collisions
  - Use Rutherford scattering cross section in stable plasma – \( \delta \phi \propto 1/r \)
  - The field around single particles is modified by collective effects (instabilities)

• To get IE collisions “self consistently” (even with a Coulomb MCC routine from Rutherford X-section) would require
  1. Grid resolution much shorter than wavelength \( (k \sim 1/\lambda_D \text{ here}) \)
  2. A macroparticle density that satisfies \( n\lambda_D^3 \) = experiment value

• (2) is particularly restrictive (and important) for IE collisions
Conclusions

• Use the kinetic (generalized) Bohm criterion with caution
  – It does not work for an arbitrary distribution function [it effectively assumes \( f_i(v_z = 0) = 0 \) and \( \partial f_e/\partial v_z|_{v_z=0} = 0 \)]
  – Don’t put your LIF data into it (you will get divergent integrals)
  – It places unphysical importance on low velocity particles

• Instead, use criterion from positive exponent velocity moments of PKE

• Ion-acoustic instabilities can be important for determining the IVDF and EVDF near the sheath of low temperature plasmas

• In two ion species plasmas, ion-ion two stream instabilities can be important
  – Since IE friction onset is so rapid and strong the system is very stiff
  – The difference in flow speeds can’t exceed the instability threshold \( V_1 - V_2 = \Delta V_c \)
  – \( V_1 - V_2 = \Delta V_c \) and the BC uniquely determines \( V_1 \) and \( V_2 \)
  – If ions are warm enough that \( \Delta V_c > |c_{s,1} - c_{s,2}| \), then no instabilities are expected and the usual \( V_1 - V_2 = |c_{s,1} - c_{s,2}| \) is appropriate