PATCHING THE PLASMA AND SHEATH SOLUTIONS

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Reference: N. Sternberg and V. Godyak,
Patching Curves

Two curves:

- \( f(x) = x^2 \)
- \( g(x) = -x^3 + 2 \)

Intersect at \( x = 1 \):

\[
f(1) = g(1)
\]

Patched curve:

- \( F(x) = \begin{cases} 
  g(x) = -x^3 + 2 & \text{if } x \leq 1 \\
  f(x) = x^2 & \text{if } x \geq 1 
\end{cases} \)

The patched function is always continuous at the patching point!

The patched function is differentiable if \( f'(1) = g'(1) \)
Patching Solutions of Differential Equations

Example: Given the initial value problem:

\[ y' = y \quad y(0) = 1 \]  

(1)

This problem has a unique solution:

\[ y(x) = e^x \]

Consider a second equation:

\[ z' = 3z \]  

(2)

Patch the solution of (1) with a solution of (2) at \( x=2 \).

Need to solve the initial value problem:

\[ z' = 3z \quad z(2) = y(2) = e^2 \]  

(3)
The general solution of (2) is \( z(x) = Ce^{3x} \), \( C \in R \)

\[
z(2) = e^2 \quad \Rightarrow \quad Ce^6 = e^2 \quad \Rightarrow \quad C = e^{-4} \quad \Rightarrow \quad z(x) = e^{3x-4}
\]

\( W \) is the patched solution: it satisfies (1) for \( x<2 \) and (3) for \( x>2 \).

\( W \) is continuous, but not differentiable at \( x=2 \).
The Symmetric Bounded Plasma Problem

Questions:

Can each region be modeled separately?

Where does the plasma end?

Where does the sheath begin?

Can the corresponding solutions be patched (connected) to approximate (replace) the solution of the plasma-wall problem?

No point is distinguished in any way!
Plasma-wall model

Continuity Equation: \((nv)' = Zn_e\)

Momentum Equation: \((nv^2)' + \frac{e}{M} n \varphi' = 0\)

Boltzmann Equation: \(\frac{n_e'}{n_e} = \frac{e}{kT_e} \varphi'\)

Poisson Equation: \(\varphi'' = -\frac{e}{\varepsilon_0} (n - n_e)\)

No ion collisions, cold ions, no heat transfer, constant ionization frequency

Boundary Conditions at the center (\(r=0\)): \(n(0) = n_e(0) = n_c\), \(v(0) = 0\), \(\varphi(0) = 0\), \(E(0) = -\varphi'(0) = 0\)

Boundary Conditions at the wall (\(r=w\)): \(n(w)v(w) = \sqrt{\frac{kT_e}{2\pi m}} n_e(w)\)
Normalized plasma-wall model

\[
y_e = \exp(-\eta)
\]

\[
(yu)' = y_e
\]

\[
(yu^2)' = y \eta'
\]

\[
\varepsilon^2 \eta'' = y - y_e
\]

\[
x = r \frac{Z}{v_s}
\]

\[
u_s = \left( \frac{kT_e}{M} \right)^{1/2}
\]

\[
\eta(x) = -\frac{e}{kT_e} \varphi(r)
\]

\[
u(x) = \frac{v(r)}{v_s}
\]

\[
y(x) = \frac{n(r)}{n_c}
\]

\[
y_e(x) = \frac{n_e(r)}{n_c}
\]

\[
\varepsilon = q_0 x_w
\]

\[
q_0 = \frac{\lambda_{D0}}{w}
\]

\[
x_w = w \frac{Z}{v_s}
\]

\[
\lambda_{D0} \quad \text{electron Debye radius at the plasma center}
\]

**Boundary condition at the center:**

\[
y(0) = 1 \quad \eta(0) = 0
\]

\[
u(0) = 0 \quad \eta'(0) = 0
\]

**Boundary condition at the wall:**

\[
y(x_w)u(x_w) = \gamma \exp(-\eta(x_w))
\]

\[
\gamma = \sqrt{\frac{M}{2\pi n}}
\]

For argon, \( \gamma = 108 \)
Normalized equations

**Plasma-wall region**

\[
y_e = \exp(-\eta)
\]

\[
(yu)' = y_e
\]

\[
(yu^2)' = y\eta'
\]

\[
\epsilon^2\eta'' = y - y_e
\]

**Plasma region**

\[
y \approx y_e
\]

\[
y = \exp(-\eta)
\]

\[
(yu)' = y
\]

\[
(yu^2)' = y\eta'
\]

\[
\text{can be solved analytically}
\]

**Sheath region**

\[
y_e \approx 0
\]

\[
(yu)' = 0
\]

\[
(yu^2)' = y\eta'
\]

\[
\epsilon^2\eta'' = y
\]

\[
\text{can be solved analytically}
\]

---

**Boundary condition at the center:**

\[
y(0) = 1 \quad \eta(0) = 0
\]

\[
u(0) = 0 \quad \eta'(0) = 0
\]

---

What is the boundary condition for the sheath model?

Ideally at the plasma boundary.

Where is the plasma boundary?
Plasma solution

(Self and Ewald, 1966)

\[
y_{pe} = y_p = \frac{1}{1+u_p^2}
\]

\[
\eta_p = \ln(1+u_p^2)
\]

\[
x = 2 \arctan(u_p) - u_p
\]

\[
\eta_p' = \frac{2u_p}{1-u_p^2}
\]

Singularity at \( u_p = 1 \) (\( v_p = v_s \))

\[
u_p(x_p) = 1 \quad y_p(x_p) = \frac{1}{2}
\]

\[
\eta_p(x_p) = \ln(2) \quad x_p = \frac{\pi}{2} - 1
\]

Is \( x_p \) the plasma boundary?

Plasma solution cannot reach the wall:

\[
y_p(x_p)u_p(x_p) \neq \gamma \exp(-\eta_p(x_p))
\]

*In the plasma: \( u<1 \) (\( v<v_s \))* (Persson 1962)

Cannot patch plasma and sheath solution at \( x=x_p \). Need \( \eta'(x_p)\neq\infty \). Try patching with \( u_p<1 \).

In Fig. curve 4 is a solution of the full model, curves 1-3 are patched solution.

![Graph comparing plasma solution to other solutions](image)
Boundary condition for the sheath models

Sheath solution needs to satisfy the boundary condition at the wall.

Idea: Solve the sheath model starting at the wall using values from the full model

Plasma and sheath solutions do not connect!

plasma-wall model (BC at the center)

\[
\frac{1}{2} \varepsilon^2 (\eta')^2 = yu^2 + e^{-\eta} - 1
\]

\[ e^{-\eta} << 1 \]

\[
\frac{1}{2} \varepsilon^2 (\eta')^2 = yu^2 - 1
\]

sheath model (BC at the wall)

connect with plasma

\[ (\eta_1')^2 \leq 0 \]

curve 1 - the plasma solution
curve 2 – sheath solution
curve 3 – plasma –wall solution
## Changing the Sheath Model

### Plasma-wall region

- \( y_e = \exp(-\eta) \)
- \((yu)' = y_e \)
- \((yu^2)' = y\eta' \)
- \( \varepsilon^2\eta'' = y - y_e \)

### Plasma-Sheath model (Langmuir, 1929)

- \( Ze^{-\eta} \approx 0 \)
- \((yu)' = 0 \)
- \((yu^2)' = y\eta' \)
- \( \varepsilon^2\eta'' = y - e^{-\eta} \)

### Sheath region

- \( y_e \approx 0 \)
- \((yu)' = 0 \)
- \((yu^2)' = y\eta' \)
- \( \varepsilon^2\eta'' = y \)

Coincide in the sheath region

Solve the plasma-sheath model starting at the wall using values from the full model (it will follow the full solution):

- \( y_{p1} = \frac{1}{1 + u_{pe}^2} = e^{-\eta_{p1}} \)

Bohm’s solution 1949

- \( \frac{1}{2} \varepsilon^2(\eta')^2 = yu^2 + e^{-\eta} - 1 \)

Near plasma boundary

- \( (\eta_p') = 0 \)
- \( (\eta_p') = \frac{2u_{p1}}{1 - u_{p1}^2} \neq 0 \)

Connect with plasma
A closer look at the Bohm solution

\[(yu)' = 0\]
\[(yu^2)' = y\eta'\]
\[\varepsilon^2\eta'' = y - y_e\]

\[\frac{1}{2}\varepsilon^2(\eta')^2 = yu^2 + e^{-\eta} - 1\]

Two solutions cannot intersect, i.e., cannot be at the same point for \(x < \infty\).

Starting at the wall move towards the plasma boundary:

\[u(x) \to u_p(x_p) = 1\]
\[y(x) \to y_p(x_p) = \frac{1}{2}\]
\[e^{-\eta(x)} \to e^{-\eta_p(x_p)} = \frac{1}{2}\]

Bohm’s solution: gives a good approximation in the sheath (near the wall), but cannot be used for patching!
A closer look at Bohm’s solution

Plasma-Sheath model:

\[ yu = \text{const.} \]
\[ u(x) \to u_p(x_p) = 1 \]
\[ y(x) \to y_p(x_p) = \frac{1}{2} \]
\[ \eta(x) \to \ln(2) \]

as \( x \to -\infty \)

\[ yu = \frac{1}{2} \Rightarrow \eta(x_w) = \ln(2\gamma) = 5.375 \]
\[ u = (2(\eta - \ln(2)) + 1)^{1/2} \Rightarrow u(x_w) = (2 \ln(\gamma) + 1)^{1/2} = 3.22 \]
\[ y(x_w) = \frac{1}{2u_w} = 0.155 \]
\[ \frac{1}{2} \varepsilon^2 (\eta')^2 = yu^2 + e^{-\eta} - 1 = \frac{1}{2}u + e^{-\eta} - 1 \]
\[ \varepsilon\eta'(x_w) = 1.109 \]

Need numerical integration!

Sheath model:

(can be solved analytically G.&S., 1990)

\[ \frac{1}{2} \varepsilon^2 (\eta')^2 = \frac{1}{2}u - 1 \]

\[ u \geq 2 \]
Other solutions of the Plasma-Sheath model

Idea: Solve the plasma-sheath model starting at a point in the plasma (index 1).

\[
\frac{1}{2} \varepsilon^2 (\eta')^2 = y_1 u_1 u + e^{-\eta} + \frac{1}{2} \varepsilon^2 (\eta_1')^2 - y_1 u_1^2 - e^{-\eta_1}
\]

\[
= \frac{u_1}{1 + u_1^2} u + e^{-\eta} + \frac{1}{2} \varepsilon^2 \left( \frac{2u_1}{1 - u_1^2} \right)^2 - 1
\]

curve 1: plasma-wall solution; curve 2 \((u_1=0.9)\) and curve 3 \((u_1=0.99)\) are patched solutions.

There must be a \(0.9<u_1<0.99\) which yields a good approximation of the plasma-wall solution

Where is the correct patching point \(u_1\) ?
Idea for finding the patching point

We know:

1) The sheath solution \((y_e \approx 0)\) approximates the plasma-wall solution in the sheath region, near the wall.

2) There is a \(u_1\) such that the solution of the plasma-sheath model is close to the plasma-wall solution and therefore to the sheath solution in the sheath.

Idea:

Choose \(u_1\) such that the solution of the plasma-sheath model connects with the solution of the sheath model.

Need to find a point on the sheath solution

We cannot use the wall (Bohm’s solution). Can we find the sheath edge?
Godyak’s criterion for the sheath edge

Godyak (1982):

Balance between the electric field energy at the sheath edge \((r=s)\) and the electron kinetic energy at the plasma boundary \((r=p)\):

\[
\frac{1}{2} \varepsilon_0 E^2(s) = \frac{1}{2} n_e(p) kT_e
\]

\[
|E(s)| = \frac{kT_e}{e \lambda_{D1}} \quad (\varepsilon \eta'(x_s) = y_p^{1/2} (x_p) = \frac{1}{\sqrt{2}})
\]

\(\lambda_{D1}\) is the electron Debye radius at the plasma boundary

\[
\lambda_D^2 = \frac{\varepsilon_0 kT_e}{e^2 n(r)} \Rightarrow \lambda_{D1} = \lambda_{D0} \sqrt{2}
\]
Deriving Godyak’s Criterion from the Boltzmann equation

Boltzmann Equation at the sheath edge $r=s$:

\[ n_e'(s) = \frac{e}{kT_e} n_e(s) \varphi'(s) \]

Discretize:

\[ n_e'(s) = \frac{n_e(s + \lambda_{D1}) - n_e(s)}{\lambda_{D1}} \approx -\frac{n_e(s)}{\lambda_{D1}} \]

At the sheath edge $r=s$:

\[ E(s) = -\varphi'(s) = -\frac{kT_e}{e \lambda_{D1}} \]

Normalized:

\[ \eta'(x_s) = y_p^{1/2}(x_p) = \frac{1}{\varepsilon \sqrt{2}} \]
Sheath solution:

\[ \frac{1}{2} \varepsilon^2 (\eta')^2 = \frac{1}{2} u - 1 \]

At the sheath edge:

\[ \eta'(x_s) = \frac{1}{\varepsilon \sqrt{2}} \]

\[ u(x_s) = 2.5 \]
Finding the patching point

Solution of plasma-sheath model with BC in the plasma:

\[ \frac{1}{2} \varepsilon^2 (\eta')^2 = \frac{u_1}{1+u_1^2} u + e^{-\eta} + \frac{1}{2} \varepsilon^2 \left( \frac{2u_1}{1+u_1^2} \right)^2 - 1 \]

\[ u(x_s) = \frac{5}{2} e^{-\eta(x_s)} \ll 0 \]

\[ \eta'(x_s) = \frac{1}{\varepsilon \sqrt{2}} \]

\[ \frac{1}{4} = \frac{5}{2} \frac{u_1}{1+u_1^2} + \frac{1}{2} \varepsilon^2 \left( \frac{2u_1}{1+u_1^2} \right)^2 - 1 \]

\[ u_1 = 1 - \left( \frac{4}{5} \right)^{1/4} \varepsilon^{1/2} \]

Note:

\[ \varepsilon = q_0 x_w = q_1 x_p y_p^{1/2} (x_p) \]

\[ \varepsilon = 0.4036q_1 \]

\[ u_1 = 1 - 0.601 q_1^{1/2} \]

\[ q_1 = 10^{-2} \]
At the patching point

Normalized coordinates

\[ \delta = 0.601 q_1^{1/2} \]
\[ u_1 = 1 - \delta \]
\[ \eta_1 = \ln(1 + u_1^2) = \ln(2) - \delta \]
\[ y_1 = \exp(-\eta_1) = \frac{1}{2} (1 + \delta) \]
\[ \eta'_1 = \frac{2u_1}{1 - u_1^2} = \frac{1}{\delta} \]
\[ x_1 = 2 \arctan(u_1) - u_1 = 0.5708 - \frac{1}{2} \delta^2 \]

Real coordinates

\[ v_1 = v_s \left( 1 - 0.6 \sqrt{\frac{\lambda_{D1}}{p}} \right) \]
\[ \varphi_1 = -\frac{k T_e}{e} \left( \ln(2) - 0.6 \sqrt{\frac{\lambda_{D1}}{p}} \right) \]
\[ n_1 = \frac{n_0}{2} \left( 1 + 0.6 \sqrt{\frac{\lambda_{D1}}{p}} \right) \]
\[ r_1 = p - 0.32 \lambda_{D1} \]
\[ \varphi'_1 = -0.95 \frac{k T_e}{e \lambda_{D1}^{1/2} p^{1/2}} \]
Results

Comparison of the plasma-wall (1) and patched (2) solutions
Results (cont.)

Comparison of the plasma-wall (1) and patched (2) solutions
Summary

Result: It is possible to patch the plasma solution with the appropriate solution of the plasma-sheath model to approximate the plasma-wall solution.

Patching solutions of two models is done in two steps:

- the first model is solved up to some end point;
- the values of the first solution at the end point are set as initial condition for the second model.

The difficulty for the plasma-wall problem is in finding the correct models, the correct solutions to patch and the correct patching point.

The plasma and sheath models, or plasma and plasma-sheath models are the obvious choices, but

- the plasma solution becomes singular and does not extend into the sheath;
- the sheath solution does not extend into the plasma region;
- the classical Bohm solution of the plasma-sheath model has an infinite domain.

Our solution of the plasma-sheath model connects a point (the patching point) of the plasma solution with the sheath edge and continues up to the wall.